

Bethe-Salpeter equation in Minkowski space with cross-ladder kernel

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A new method for solving the Bethe-Salpeter equation is developed. It allows to find the Bethe-Salpeter amplitudes both in Minkowski and in Euclidean spaces and, as a by product, the light-front wave function. The method is valid for any kernel given by irreducible Feynman graphs. Bethe-Salpeter and Light-Front equations for scalar particles with ladder + cross-ladder kernel are solved.

1. INTRODUCTION

Bethe-Salpeter (BS) equation [1] is an important tool for studying the relativistic bound state problem in a field theory framework (see for review [2]). For a bound state with total momentum p and in case of equal mass particles, it reads

$$\Phi(k, p) = G_0^{(12)} \int \frac{d^4 k'}{(2\pi)^4} iK(k, k', p) \Phi(k', p), \quad (1)$$

where Φ is the BS amplitude, K the interaction kernel, m the mass of the constituents, k their relative momentum and $G_0^{(12)}$ the free propagators of the constituent particles:

$$\begin{aligned} G_0^{(12)}(p, k) &= G_0^{(1)}(p, k) G_0^{(2)}(p, k) = \\ &= \frac{i}{(\frac{p}{2} + k)^2 - m^2 + i\epsilon} \frac{i}{(\frac{p}{2} - k)^2 - m^2 + i\epsilon}. \end{aligned} \quad (2)$$

We will denote by $M = \sqrt{p^2}$ the total mass of the bound state, and by $B = 2m - M$ its binding energy.

The BS equation has singularities which make difficult to find its numerical solution. These singularities are due to the free propagators (2) but can also result from the interaction kernel itself. To overcome this difficulty, Wick [3] formulated the BS equation in the Euclidean space, by rotating the relative energy k_0 in the complex plane,

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i.e., by introducing the variable $k_4 = -ik_0$. This leads to a well defined integral equation – see (12) below – which can be solved by standard methods and provides the mass of the system. In this procedure the original BS amplitude is however lost, and the "rotated" one can no longer be used in calculating other physical observables, like for instance form factors.

A successful attempt to calculate the BS amplitude in Minkowski space for the ladder kernel was presented in [5]. It was based on the Nakanishi integral representation of the amplitude [4]. Another approach to solve BS equation with separable interaction in Minkowski space was developed in [6] and applied to the nucleon-nucleon system. In [7], an equation obtained by projecting the BS equation on the light-front (LF) plane was studied: the LF kernel was found approximately, as an expansion in terms of the BS one and the original BS amplitude was not reconstructed from its LF projection.

We present in this paper – based on our works [8,9] – a new method for solving the BS equation. This method provides the BS amplitude both in Minkowski and in Euclidean space, depending on the value (real or imaginary) of the k_0 variable. Our approach is based on a projection of the BS equation on the LF plane, and on the Nakanishi integral representation [4] of the BS amplitude. The LF projection plays the role of an integral

transform which removes the singularities of the BS amplitude. The Nakanishi representation results in an equation in a closed form and allows to easily restore with its solution, the original BS amplitude. The transformed equation is derived without any approximation and remains equivalent to the original BS one. Our method is not restricted to the ladder kernel. For more complicated interactions, *e.g.* cross box, calculations become more lengthy, but the additional difficulties are due to the evaluation of the Feynman diagram for the kernel itself and not to the solution of the equation.

In order to present the method more distinctly, we consider the case of zero total angular momentum and spinless particles.

In sect. 2 we derive the equation allowing to find the BS amplitude both in Minkowski and Euclidean spaces. In sect. 3 we present corresponding LF equation and in sect. 4 – the BS equation in Euclidean space. In sect. 5 the numerical solutions are presented. Sect. 6 contains some concluding remarks.

2. BETHE-SALPETER EQUATION ON THE LIGHT-FRONT PLANE

Our method is inspired by an existing relation – see eq. (7) below – between the (singular) BS amplitude $\Phi(k, p)$ and the (non-singular) two-body LF wave function $\psi(\vec{k}_\perp, x)$. The latter can be obtained by projecting the BS amplitude on the LF plane. We use the covariant formulation [10] with the LF plane, defined by $\omega \cdot x = 0$ with $\omega^2 = 0$. The particular choice $\omega = (\omega_0, \vec{\omega}) = (1, 0, 0, -1)$ results in the standard LF form $t + z = 0$. We apply to both sides of (1) the integral transform:

$$\int_{-\infty}^{\infty} d\beta \Phi(k + \beta\omega, p) = \int_{-\infty}^{\infty} d\beta G_0^{(12)}(k + \beta\omega, p) \times \int \frac{d^4 k'}{(2\pi)^4} iK(k + \beta\omega, k', p) \Phi(k', p), \quad (3)$$

In the transformed equation, the BS amplitude is written in terms of the Nakanishi integral representation [2], which for zero angular momentum reads:

$$\Phi(k, p) = \frac{-i}{\sqrt{4\pi}} \int_{-1}^1 dz' \int_0^\infty d\gamma' \quad (4)$$

$$\times \frac{g(\gamma', z')}{[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p \cdot k z' - i\epsilon]^3}.$$

A similar representation exists for non-zero angular momentum. This representation is valid for a rather wide class of solutions. This leads to the following equation, derived in [8], for the weight function $g(\gamma, z)$:

$$\int_0^\infty \frac{g(\gamma', z) d\gamma'}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2) \kappa^2]^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz' V(\gamma, z; \gamma', z') g(\gamma', z'), \quad (5)$$

where V is a kernel given in terms of the BS interaction kernel K by

$$V(\gamma, z; \gamma', z') = \int_{-\infty}^{\infty} d\beta G_0^{(1)}(p, k + \beta\omega) G_0^{(2)}(p, k + \beta\omega) \times \frac{\omega \cdot p}{2^4 \pi^5} \int \frac{K(k + \beta\omega, k', p) d^4 k'}{[k'^2 + p \cdot k' z' - \gamma' - \kappa^2 + i\epsilon]^3}. \quad (6)$$

$G_0^{(i)}$ are the free propagators (2), the denominator arises from the representation (4), the integration over k' results from the BS equation (1) and the integration over β from the LF projection (3). The bound state mass M enters through the parameter $\kappa^2 = m^2 - \frac{1}{4}M^2$.

The four-vectors in the r.h.s of eq. (6) appear in form of scalar products, expressed through the equation variables (γ, z) defined by:

$$k^2 = -\frac{(\gamma + z^2 m^2)}{1 - z^2}, \quad \frac{\omega \cdot k}{\omega \cdot p} = -\frac{1}{2}z, \\ p \cdot k = \frac{z[\gamma + z^2 m^2 + (1 - z^2) \kappa^2]}{1 - z^2},$$

and related to the standard LF variables by $\gamma = k_\perp^2, z = 1 - 2x$.

Equation (5) is equivalent to the initial BS equation (1) and provides, for a given kernel K , the same bound state mass M . Once $g(\gamma, z)$ is known, the BS amplitude can be restored by eq. (4). For real values of k_0 , we find the BS amplitude in Minkowski space and for imaginary values $k_0 = ik_4$, in the Euclidean one. The corresponding LF wave function $\psi(k_\perp, x)$ is easily obtained

by

$$\begin{aligned} \psi(k_\perp, x) = & \frac{x(1-x)}{\pi} \int_{-\infty}^{\infty} \Phi(k + \beta\omega, p)(\omega \cdot p) d\beta = \\ & \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \frac{x(1-x)g(\gamma', 1-2x)d\gamma'}{[\gamma' + k_\perp^2 + m^2 - x(1-x)M^2]^2}. \end{aligned} \quad (7)$$

2.1. Ladder kernel

Application to the ladder BS kernel

$$iK^{(L)}(k, k', p) = \frac{i(-ig)^2}{(k - k')^2 - \mu^2 + i\epsilon} \quad (8)$$

allows to test our method. In this case, the kernel $V = V^{(L)}$ of equation (5) is calculated analytically.

It becomes especially simple for $\mu = 0$. This particular case constitutes the Wick-Cutkosky model [3,11]. We search the solution of (5) in the form:

$$g(\gamma, z) = \delta(\gamma) g(z)$$

justified by possibility to find a function $g(z)$ which does not depend on γ . The integration over γ' in both sides of equation (5) drops out, the γ -dependence of the kernel V is reduced to a factor which is the same in both sides of equation and cancels. After that, the equation (5) takes a simplified form

$$g(z) = \frac{\alpha}{2\pi} \int_{-1}^1 dz' \tilde{V}(z, z') g(z') \quad (9)$$

with $\alpha = g^2/(16\pi m^2)$ and

$$\begin{aligned} \tilde{V}(z, z') &= \frac{m^2}{m^2 - \frac{1}{4}(1 - z'^2)M^2} \\ &\times \begin{cases} \frac{(1-z)}{(1-z')}, & \text{if } -1 \leq z' \leq z \leq 1 \\ \frac{(1+z)}{(1+z')}, & \text{if } -1 \leq z \leq z' \leq 1 \end{cases} \end{aligned}$$

which exactly coincides with the Wick-Cutkosky equation [2,3,11].

Numerical results for non-zero μ are given in sect. 5.

2.2. Cross ladder kernel

The cross-ladder BS kernel is shown in Figure 1. The corresponding amplitude reads:

$$K^{(CL)}(k, k', p) = \frac{-ig^4}{(2\pi)^4} \quad (10)$$

$$\begin{aligned} &\times \int \frac{1}{[p''^2 - m^2 + i\epsilon][(p'_2 - p_1 + p'')^2 - m^2 + i\epsilon]} \\ &\times \frac{d^4 p''}{[(p_1 - p'')^2 - \mu^2 + i\epsilon][(p'_1 - p'')^2 - \mu^2 + i\epsilon]}. \end{aligned}$$

We calculate this expression, substitute the result in (6), calculate the integrals and find in this way the cross-ladder contribution to the kernel $V(\gamma, z; \gamma', z')$ in equation (5). The full kernel is the sum of ladder and cross-ladder graphs:

$$V = V^{(L)} + V^{(CL)}.$$

The details of calculation and expression for $V^{(CL)}$ are given in [9].

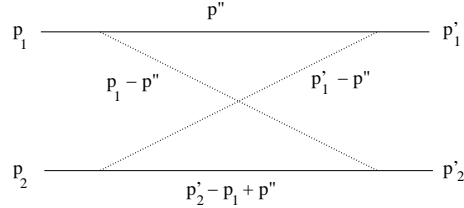


Figure 1. Feynman cross graph.

3. LIGHT-FRONT EQUATION AND KERNELS

We would like to compare the results obtained in the BS approach with the ones found in Light-Front Dynamics (LFD). With this aim, we precise here the LF equation and the corresponding kernel. In terms of \vec{k}_\perp and x variables, the LF equation reads (see *e.g.* [10]):

$$\begin{aligned} &\left(\frac{\vec{k}_\perp^2 + m^2}{x(1-x)} - M^2 \right) \psi(\vec{k}_\perp, x) = -\frac{m^2}{2\pi^3} \\ &\times \int V_{LF}(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2) \frac{\psi(\vec{k}'_\perp, x') d^2 k'_\perp dx'}{2x'(1-x')}. \end{aligned} \quad (11)$$

There are two time-ordered ladder graphs and six LF time-ordered cross-ladder graphs. The latter have the order α^2 . In addition, and to the same order α^2 , there are two irreducible time-ordered graphs with two mesons in the intermediate state (stretched boxes). The full LFD kernel –

including ladder, cross-ladder and stretched-box graphs – is written in the form:

$$V_{LF}(\vec{k}'_{\perp}, x'; \vec{k}_{\perp}, x, M^2) = V_{LF}^{(L)} + V_{LF}^{(CL)} + V_{LF}^{(SB)}.$$

The LF equation (11) has been solved with this full kernel.

4. CROSS LADDER IN EUCLIDEAN SPACE

The possibility of Wick rotation has been proved [3] for the ladder kernel. One can also "rotate", without crossing singularities, simultaneously all the energies k_{i0} of a perturbative Feynman amplitude. However, this possibility is not evident for a BS amplitude with a kernel more complicated than the ladder one, since now not all the energies are rotated: when the relative energy k_0 is replaced by ik_4 , the eigenvalue – the bound state energy p_0 – still remains real. For the latter case, the validity/invalidity of Wick rotation is discussed in the literature. For the cross-ladder kernel, we can check it numerically, by finding the binding energies from eq. (5) – see results on Table 2 below – and by solving the Euclidean space equation for L+CL kernel. One can then see whether or not the Euclidean and Minkowski results coincide with each other. Coincidence would confirm both: the validity of Wick rotation for the cross ladder kernel and our method in Minkowski space.

Equation in Euclidean space is obtained from (1) by the replacement: $k_0 = ik_4$, $k'_0 = ik'_4$. In the rest frame $\vec{p} = 0$ it reads:

$$\left[\left(k_4^2 + \vec{k}^2 + m^2 - \frac{M^2}{4} \right)^2 + M^2 k_4^2 \right] \Phi_E(k_4, k) = \int \frac{dk'_4 d^3 k'}{(2\pi)^4} K_E(k_4, \vec{k}; k'_4, \vec{k}') \Phi_E(k'_4, k'), \quad (12)$$

where $K_E(k_4, \vec{k}; k'_4, \vec{k}') = K(ik_4, \vec{k}; ik'_4, \vec{k}')$ and $\Phi_E(k_4, k) = \Phi(ik_4, k)$. We take sum of ladder, eq. (8), and cross ladder, eq. (10), "convert" them in Euclidean space and find in this way the kernel $K_E = K_E^{(L)} + K_E^{(CL)}$ of the equation (12).

5. NUMERICAL RESULTS

It turns out that the discretized integral operator in l.h.-side of eq. (5) has very small eigenvalues. They are unphysical but make the solution unstable. We have regularize it by adding a small constant ϵ to its diagonal part. This procedure allows us to obtain stable eigenvalues in the interval $\epsilon = 10^{-4} \div 10^{-12}$.

Table 1

Coupling constant $\alpha = g^2/(16\pi m^2)$ as a function of the binding energy B for ladder kernel with $\mu = 0.15$ and $\mu = 0.5$ obtained with $\epsilon = 10^{-6}$.

B	$\alpha(\mu = 0.15)$	$\alpha(\mu = 0.50)$
0.01	0.5716	1.440
0.10	1.437	2.498
0.20	2.100	3.251
0.50	3.611	4.901
1.00	5.315	6.712

For the ladder kernel with $\mu = 0.15$ and $\mu = 0.5$ and unit constituent mass $m = 1$ the values of binding energy $B = 2m - M$ are displayed in Table 1. With all shown digits, they are in full agreement with the results of [12] obtained by using the Wick rotation and the method of [13]. This demonstrates the validity of our approach.

The corresponding weight function g for a system with $\mu = 0.5$ and $B = 1.0$ is plotted in Figure 2. Its γ -dependence is not monotonous and has a nodal structure; the z -variation is also non trivial. We have remarked a strong dependence of $g(\gamma, z)$ relative to values of the ϵ parameter smaller than $\sim 10^{-4}$, in contrast to high stability of corresponding eigenvalues. However, the corresponding BS amplitude Φ and LF wave function ψ , obtained from $g(\gamma, z)$ by the integrals (4) and (7), show the same strong stability as the eigenvalues.

The ladder BS amplitude at the rest frame $\vec{p} = 0$ is shown in Figure 3. On the left, we have plotted $\Phi(k_0, k)$ in Minkowski space versus k_0 . It exhibits a singular behaviour due to the poles of

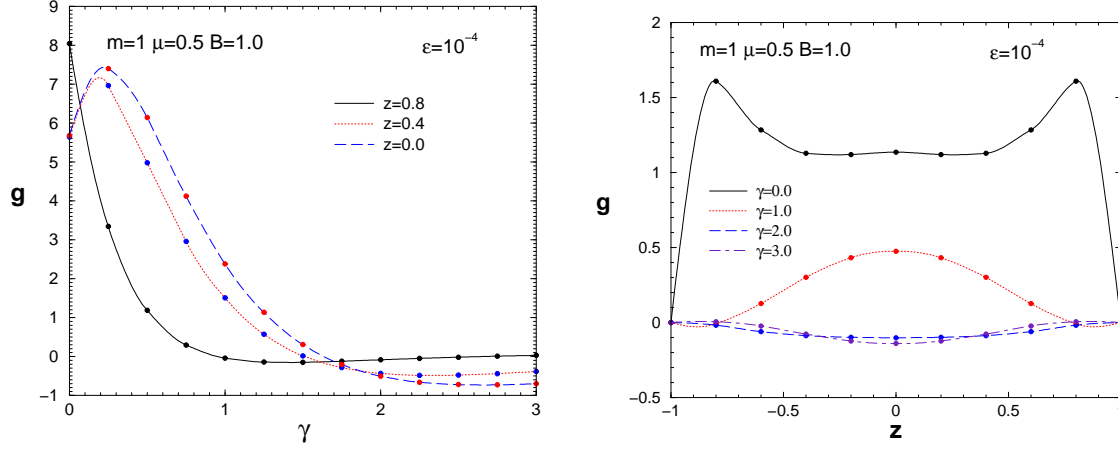


Figure 2. Function $g(\gamma, z)$ for ladder kernel for $\mu = 0.5$ and $B = 1.0$. On left – versus γ for fixed values of z and on right – versus z for a few fixed values of γ .

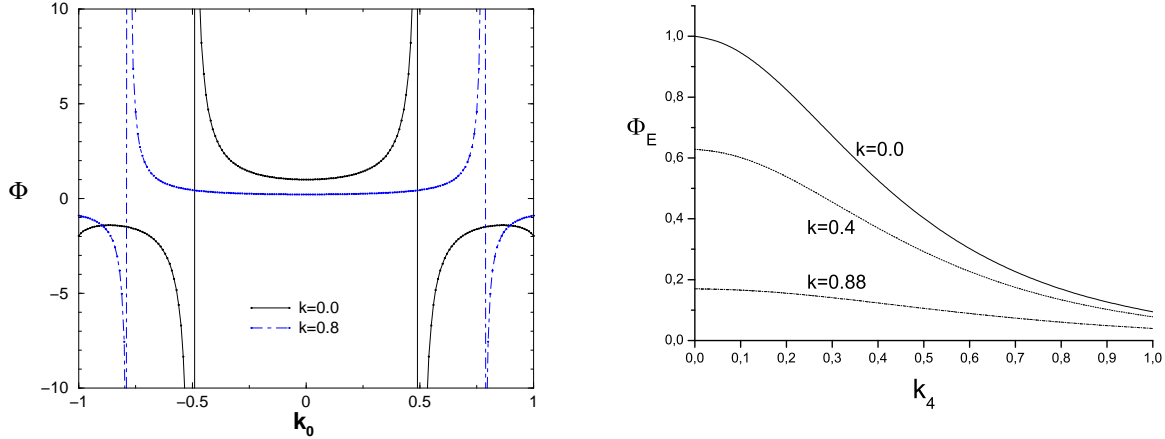


Figure 3. BS amplitude for ladder kernel with $\mu = 0.5$, $B = 1.0$ and fixed values of k . On left, $\Phi(k_0, k)$ in Minkowski space versus k_0 ; on right $\Phi_E(k_4, k) = \Phi(k_0 = ik_4, k)$ in Euclidean space versus k_4 .

the propagators in (1) at $k_0 = \pm (\varepsilon_k \pm \frac{M}{2})$, *i.e.* poles moving with \vec{k} and M . On the right, we have plotted $\Phi_E(k_4, k) = \Phi(k_0 = ik_4, k)$ in Euclidean space versus k_4 . Both are calculated by eq. (4) with the substitution $k_0 = ik_4$ for the Euclidean BS amplitude $\Phi_E(k_4, k)$. They drastically differ from each other though they correspond to the same binding energy. The Euclidean solution is smooth, in contrast to the Minkowski one. The Euclidean amplitude found by direct solution of the Wick-rotated equation (12) is indistinguish-

able from the one shown at r.h.-side of Figure 3. The corresponding LF wave function $\psi(k_\perp, x)$ is shown in Figure 4.

The cross ladder effects are presented in Figure 5, where the binding energy B as a function of the coupling constant α is shown for exchange masses $\mu = 0.5$. Corresponding numerical values – with an accuracy of 1% – are given in Table 2. We see that for the same kernel – ladder or ladder + cross-ladder – BS and LFD binding energies are very close to each other. The BS equation is

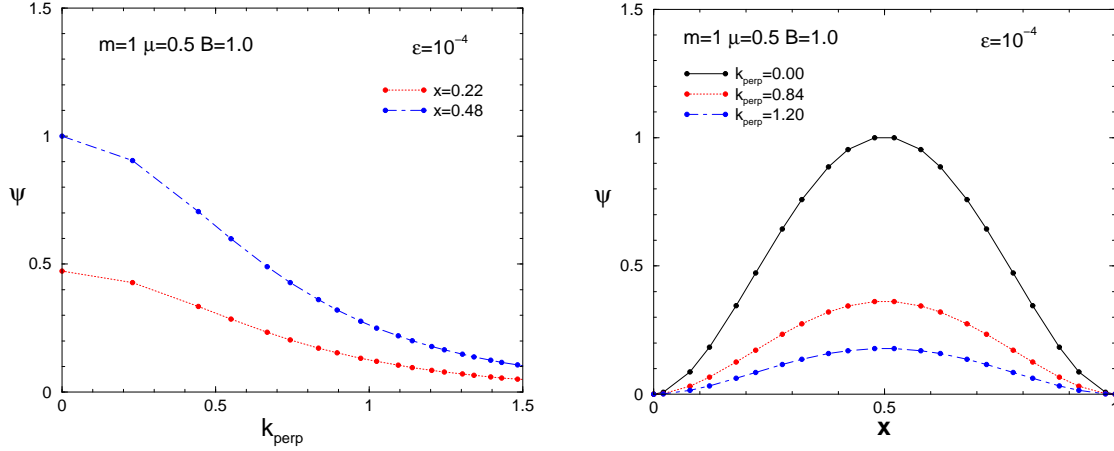


Figure 4. Wave function $\psi(k_{\perp}, x)$ for ladder kernel with $\mu = 0.5$ and $B = 1.0$. On left versus k_{\perp} for fixed values of x and on right versus x for a few fixed values of k_{\perp} .

Table 2

Coupling constant α for given values of the binding energy B and exchanged mass $\mu = 0.5$ calculated with BS and LF equations for the ladder (L), ladder +cross-ladder (L+CL) and (in LFD) for the ladder +cross-ladder +stretched-box (L+CL+SB) kernels.

B	BS(L)	BS(L+CL)	LF(L)	LFD(L+CL)	LFD(L+CL+SB)
0.01	1.44	1.21	1.46	1.23	1.21
0.05	2.01	1.62	2.06	1.65	1.62
0.10	2.50	1.93	2.57	2.01	1.97
0.20	3.25	2.42	3.37	2.53	2.47
0.50	4.90	3.47	5.16	3.67	3.61
1.00	6.71	4.56	7.17	4.97	4.91

slightly more attractive than LFD. At the same time, the results for ladder and ladder+cross-ladder kernels considerably differ from each other. The effect of the cross-ladder is strongly attractive. Though the stretched box graphs are included, their contribution to the binding energy is smaller than 2% and also attractive. This agrees with the direct calculation of the stretched box contribution to the kernel performed in [14]. The solution of the Wick rotated equation (12) for L+CL kernel was found in [15]. Within a numerical accuracy of 1%, it coincides with the corresponding solution given by our method, *i.e.*, with the column BS(L+CL) of the Table 2. This coincidence proves the possibility to solve BS equation with cross ladder kernel by using the Wick rota-

tion. It also confirms the validity of our method. We emphasize again the fact that the Wick rotated equation allows to find the binding energy, the same as the one determined by the Minkowski BS equation, but not the Minkowski BS amplitude.

6. CONCLUSION

We have developed a new method for solving the Bethe-Salpeter equation in the Minkowski space, *i.e.* without making use of the Wick rotation. This method is based on an integral transform of the original BS equation which removes its singularities.

Our equation is more easily formulated in terms

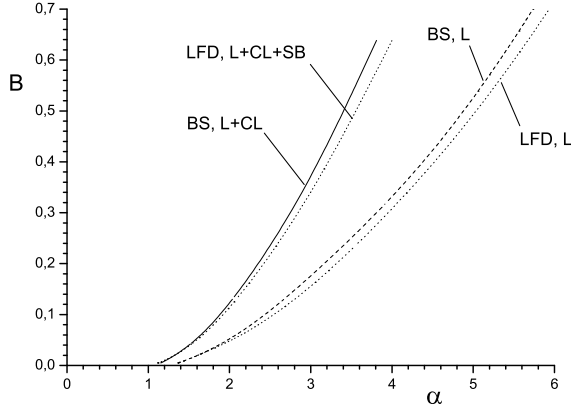


Figure 5. Binding energy B versus coupling constant α for BS and LFD equations with ladder (L), ladder+cross-ladder (L+CL), and ladder+cross-ladder+stretched box (L+CL+SB) kernels. The exchange mass is $\mu = 0.5$.

of the weight function of the Nakanishi integral representation [4]. This allows a straightforward reconstruction of the original Bethe-Salpeter amplitude – both in Minkowski and Euclidean space – as well as the Light-Front wave function.

For zero-mass ladder exchange, we reproduce analytically the Wick-Cutkosky model. For massive ladder exchange, the binding energies are found numerically and are in full agreement with preceding results obtained in the Euclidean space.

For a given kernel, Bethe-Salpeter and Light-Front approaches give very close results, though the first one is always slightly more attractive. The cross-ladder contribution is strongly attractive in both models. The stretched-box contribution – evaluated separately only in the Light-Front framework – is attractive but small.

Binding energies obtained by solving Bethe-Salpeter equation in Minkowski space with ladder plus cross-ladder kernel are also in full agreement with the ones obtained in Euclidean space [15].

We are aware about only one work [16] where the separated effect of the cross-ladder diagrams in the Bethe-Salpeter equation has been estimated. The authors used an approximate dispersion relation to obtain the corresponding kernel. Our results are found to be three times smaller

than those given in this reference.

Our method for solving the Bethe-Salpeter equation in the Minkowski space can be generalized to non-zero angular momentum and, presumably, to the fermion case.

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